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ABSTRACT

Use of low-stress algorithms to reduce the cognitive load on students is advocated. The low-stress algorithm for addition developed by Hutchings is detailed first. Then a variation on the usual algorithm is proposed: adding from left to right, writing the partial sum for each stage. Next, a "quick addition" method for adding fractions proposed by Spitler is illustrated. Attention is then turned to subtraction, with the conventional decomposition algorithm shown, followed by Hutchings' low-stress algorithm. Then an algorithm proposed by Brown, involving no "borrowing," is presented. It is suggested that alternative algorithms should be tried with students who exhibit problems using conventional algorithms, and also for enrichment with other students. (MNS)

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ARE LOW STRESS ALGORITHMS IN ADDITION AND SUBTRACTION A SOLUTION TO MORE THAN MATH PROBLEMS?

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ARE LOW-STRESS ALGORITHMS IN ADDITION AND SUBTRACTION A SOLUTION TO MORE THAN MATH PROBLEMS?

An algorithm, simply stated, is the step-by-step procedure for completing a mathematical process of computation. Many algorithms for performing the basic operations of addition, subtraction, multiplication, and division of whole numbers have been developed and are available for the classroom teacher. Yet, in our present curriculum, few students have the opportunity to examine alternative ways of performing an operation.

In the 1976 NCTM Yearbook, Measurement in School Mathematics, Barton Hutchings presented a series of low-stress algorithms that could be used as alternatives to the conventional ones that are taught in our elementary schools. A survey of NCTM's publications, Arithmetic Teacher and Mathematics Teachers, show that there was an increased interest in the development of alternative algorithms in addition and subtraction in the years immediately following this essay (Brown, 1982; Colton, 1980; Hutchings, 1975; Spitler, 1979; Vance, 1982). The general thesis for these algorithms lies in the claim that they are performed with less stress than conventional algorithms, thereby giving the learner greater computational power. This claim is especially important for children who experience difficulty in using conventional algorithms.

Although NCTM and other professional organizations suggest the use of calculators and computers in elementary mathematics, at no point is the need for computational skills to be

neglected. In a world that is increasingly quantitative in orientation, children must master more mathematics than at any other time in history. Unfortunately, there is no extra time for learning it. Hutchings argues that this "great increase in mathematical concepts and generalizations in the curriculum is fundamentally in conflict with the large amount of time and energy required for the mastery of conventional computation algorithms" (Hutchings, 1976).

Children who experience learning problems may have difficulty performing several subtasks simultaneously. For example, in the addition of a column of numbers, the student has to add intermediate values, keep track of these values mentally and often "carry" into the next column to be added. The decomposition algorithm in subtraction requires comparing digits, borrowing or regrouping often over zeros, and the application of difficult subtraction facts.

By reducing the cognitive load required for intermediate tasks, one can argue that the cognitive capacity would be increased for other tasks. This decrease in cognitive load would result in less stress for the student (Dashiell & Killian, 1981). Alternate algorithms may decrease cognitive load in two distinct ways:

1. Supplementary notation is often used to record every step, thereby, reducing the need to perform intermediate steps mentally.
2. The learner can usually complete intermediate steps of a distinct kind rather than alternate between different kinds of intermediate steps.

Instructional, psychological, and computational advantages derive from characteristics such as these. Not only do the algorithms allow computational work with a minimum of stress, they also facilitate identification of specific errors and the analysis of error patterns. Although low-stress algorithms do not reduce the number of steps involved, they may change the order in which the steps are completed so that the student completes all the steps of one process before he moves on to the next.

Special attention should be paid to these algorithms by teachers of children in the middle grades who have not mastered the conventional algorithms. Each of these algorithms can be mastered readily after brief training.

Low Stress Addition

Barton Hutching's low-stress algorithm for addition uses a form of notation called half-space notation. Half-space notation uses numbers no more than a half-space in height to record the sum of two digits at each intermediate step. The unit portion of the sum is written at the lower right of two addends, the tens portion is written at the lower left. In half-space notation, the line separating the addends from their sum is omitted in the basic facts, so that the slower child is not tempted to repeat the line or operation sign when doing column addition. Therefore,

6	9	8	6
+ 7	+ 6	+ 4	+ 3
13	15	12	9

would be written:

6	9	8	6
$_1{}^7{}_3$	$_1{}^6{}_5$	$_1{}^4{}_2$	$_{}{}^3{}_9$

The low-stress algorithm is particularly useful for column addition. As in all low-stress algorithms, a complete record of each immediate step is made. Therefore, every basic fact necessary for the completion of the algorithm is recorded rather than recalled from memory. These mental calculations are a major problem for many students using conventional algorithms.

In the example below, the basic fact for each set of addends has its sum recorded as the addition is completed, eliminating the mental accumulation of sums.

9	Add 9 + 6, record 15 (half-space notation).
$_1{}^6{}_5$	
$_1{}^7{}_2$	Add 5 + 7, record 12.
$_1{}^8{}_0$	Add 2 + 8, record 10.
$_{}{}^7{}_7$	Add 0 + 7, record 7.
+ $_1{}^6{}_3$	Add 7 + 6, record 13.

Notice that it is after the first step that the procedure differs from the procedure for basic facts: the sum 15 is not used in the next step. Instead, only the 5 or units portion is used as one addend in the second set of addends. Continue in this manner, finding each fact until the column is completed.

The sum of the column is determined by recording the one's portion of the last sum below the line. Count the number of times a ten was recorded at the left of the column and record this.

$_{1}6_{5}$

$_{1}7_{2}$

$_{1}8_{0}$

7_{7}

\+ $_{1}6_{3}$ Record the 3 below the line.

---

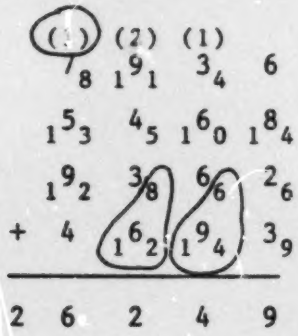
4 3 Count the number of times a ten was recorded (4 times). Record this number in the tens place.

Addition involving multi-columns will proceed in a like manner. Remember to space the columns further apart to allow for the special notation. Regrouping is done as in the usual, algorithm by counting the number of the times a ten was recorded and carrying this to the top of the tens column. This value is then used as the first digit in the addition of the tens column.

Once a student has mastered the process with smaller numbers, he or she can quickly proceed to very large numbers with many addends. Consider these examples:

(2)	(2) (2)	(3) (2) (2) (1)
$_{1}9_{1}$ 6	2_{4} 6_{8} 4	4_{7} 6_{8} $_{1}8_{0}$ 5_{6} 3
3_{4} $_{1}7_{3}$	$_{1}9_{3}$ $_{1}5_{3}$ $_{1}6_{0}$	$_{1}9_{6}$ $_{1}6_{4}$ 7_{7} 3_{9} 6_{9}
4_{8} 3_{6}	$_{1}8_{1}$ 3_{6} 2_{2}	$_{1}8_{4}$ $_{1}9_{3}$ 0_{7} $_{1}3_{2}$ $_{1}1_{0}$
$_{1}2_{0}$ $_{1}9_{5}$	6_{7} $_{1}4_{0}$ 7_{9}	$_{1}9_{3}$ 6_{9} 2_{9} $_{1}8_{0}$ 4_{4}
\+ 5_{5} 2_{7}	\+ $_{1}5_{2}$ 2_{2} $_{1}3_{2}$	\+ 2_{5} $_{1}1_{0}$ $_{1}6_{5}$ 9_{9} 3_{7}
---	---	---
2 5 7	3 2 2 2	3 5 0 5 9 7

Specific error patterns can be easily located and remediated. In the following, the sum of 6 + 9 is not 14, but 15. The sum of 8 + 6 is not 12, but 14. And the 1 was carried incorrectly from the hundreds column to the thousands column.



Addition

Hutching's low-stress algorithm for addition is very useful, but may not be the algorithm needed to clarify a concept. For the child having problems with place value, the following algorithm may be explored.

The usual algorithm for adding a column of numbers involves adding the columns from right to left and regrouping accordingly. A variation of this algorithm might be to add from left to right. This addition procedure will involve more steps than the conventional method, but may facilitate a better understanding of place value.

```
  6335
+ 3263
  9000
   500
    90
     8
  9598
```

```
  3695
+ 8462
 11000
  1000
   150
     7
 12157
```

```
  6347
  4962
  3463
+ 4896
 17000
  2400
   250
    18
 19668
```

Addition of Fractions

If the conventional algorithm for addition of fractions causes trouble, the "quick addition" method proposed by Gail Spitler (1979) might be used. In this method, the student is not required to find the lowest common denominator, thus eliminating much of the stress many children experience.

Consider this example:

$$= \frac{(1 \times 3) + (1 \times 4)}{4 \times 3} = \frac{3 + 4}{12} = \frac{7}{12}$$

First multiply the denominators. Then, cross multiply to find the two addends for the numerator. Thus, we renamed 1/4 as 3/12 and 1/3 as 4/12. This renaming is the same procedure that is followed to find a common denominator for two fractions, yet eliminates much of the mental juggling that is sometimes necessary in the conventional algorithm where a student may be required to do quite a bit of work to find the lowest common denominator. Some children may find it simpler to reduce the fraction after obtaining an answer.

In the second example, the resulting denominator is not the lowest common denominator. Yet, any common denominator will suffice.

$$\frac{1}{2} + \frac{3}{8} = \frac{(1 \times 8) + (3 \times 2)}{2 \times 8} = \frac{8 + 6}{16} = \frac{14}{16} = \frac{7}{8}$$

This algorithm may seem inefficient, especially if the lowest common denominator is obvious or if more than two addends are present. Yet, by suppling our students with an alternative algorithm, stress may be relieved for many students and success achieved.

Subtraction

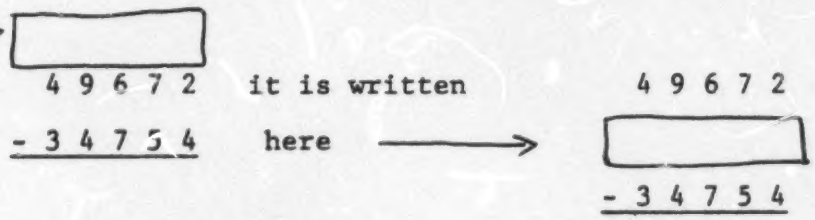
Two ways to compute a difference are described here and each can be used to decrease stress when subtracting. Before discussing specific characteristics, the conventional algorithm will be presented.

	variation	
	8 1 6 1	8(16)6(12)
4 9 6 7 2	4 9 6 7 2	4 9 6 7 2
- 3 4 7 5 4	- 3 4 7 5 4	- 3 4 7 5 4
1 4 9 1 8	1 4 9 1 8	1 4 9 1 8

The decomposition technique for subtracting is well known and will not be considered further. Instead, alternate algorithms will be examined.

Low-stress Subtraction

Low-stress subtraction involves the use of half-space notation to express regrouping. Thus, 754 can be written as $74^{1}4$ or $6^{1}54$ or $6^{1}4^{1}4$. This regrouping is written directly above the subtrahend, which helps some children in reading and organizing their work. Rather than writing the regrouped minuend here→ ☐



When doing subtraction, all regrouping is completed before any subtraction occurs and is placed between the minuend and subtrahend. The series of steps involved for the above problem are presented.

(a) 4 9 6 7 2
- 3 4 7 5 4

(b) 4 9 6 7 2
$6^{1}2$
- 3 4 7 5 4

(c) 4 9 6 7 2
$4\ 8^{1}6\ 6^{1}2$
- 3 4 7 5 4

(d) 4 9 6 7 2
$4\ 8^{1}6\ 6^{1}2$
- 3 4 7 5 4
1 4 9 1 8

Since this algorithm is a decomposition method, it is conceptually consistent with the conventional algorithm. However, performance in computation is simplified.

Subtraction with zeros in the minuend can be accomplished with greatly reduced cognitive load using the low-stress algorithm. A severely remedial child may not be able to appreciate all of the underlying concepts in the following conventional example.

8 9 9 15 1
6 9 Ø Ø Ø 1
- 1 3 9 6 7 2

Yet, he could begin renaming in the usual low-stress way.

6 9 0 0 6 1
$6\ 8\quad {}^{1}5^{1}1$
- 1 3 9 6 7 2

The zeros are ignored or skipped at this point, and the renaming continued with the first number that allows "borrowing".

After the renaming is completed the "skipped" zeros are replaced by nines in the renamed minuend. Thus, the new minuend becomes:

$$\begin{array}{r} 6\,9\,0\,0\,6\,1 \\ 6\,8\,9\,9\,{}^{1}5\,{}^{1}1 \\ -\,1\,3\,9\,6\,7\,2 \\ \hline 5\,5\,0\,3\,8\,9 \end{array}$$

In solving the problem, a student might verbalized the steps as follows:

> Bring down the 1.
> Can I take 2 from 1?
> No.
> I can borrow from the 6. That makes 11 (write the superscript 1) and leaves 5 (write the 5).
> Can I take 7 from 5?
> No.
> I can borrow from the 9. That makes 15 (write the superscript 1) and leaves 8 (write the 8).
> Can I take 3 from 8?
> Yes.
> Can I take 1 from 6?
> Yes.
> Now, I skipped two places, so I'll put in two 9s.
> Now,
> 2 from 11 is 9
> 7 from 15 is 8
> 6 from 9 is 3
> 9 from 9 is 0
> 3 from 8 is 5
> 1 from 6 is 5

This procedure requires minimal conceptual demand. All of one part of the work, the regrouping, is completed before the next part, the subtraction is done. The possibility of error is reduced, because the child is not required to shift thinking about regrouping, to subtracting, back to regrouping, back to subtracting, and so on.

Another advantage is that error detection is easier. The conventional decomposition algorithm requires verbalization of student thought or diagnostic testing in order to spot specific errors. In this low-stress method, by examining a child's work, the teacher can easily spot errors since there is a complete work record. Thus, in the following example,

6 9 3 0 6	the student has regrouped to make 12
6 $9^{1}2^{1}$0 6	hundreds, but failed to change the 9
- 5 6 9 8 2	thousands to 8 thousands.

"No Borrow" Subtraction

Brown (1982) proposed the following algorithm as an alternative to the low-stress method of Hutchings. In this technique, the traditional borrowing process is eliminated, but a double subtraction becomes necessary. This algorithm is, in effect, a variation of the equal addends method.

If an algorithm is truly low-stress, it must eliminate the conditions that cause failure in other algorithms. This technique does this in three ways:

1.) the traditional borrowing process is eliminated.
2.) students are not required to perform the steps in a prescribed order, from right to left.
3.) subtraction involves single-digit numbers from single-digit numbers.

Phase 1

The notation involved in this algorithm should be introduced and practiced before subtraction begins. This notation is simple, depending only on the recognition of the relationship between corresdponding digits in the minuend and

subtrahend and the entry of a zero or one.

Consider the following example:

(a) $4\ 3\ 9\ 2$
$-\ 3\ 6\ 4\ 6$

(b) $4\ 3\ 9\ 2^{0}$
$-\ 3\ 6\ 4\ 6$

(c) $4\ 3\ 9^{1}2^{0}$
$-\ 3\ 6\ 4\ 6$

(d) $4\ 3^{0}9^{1}2^{0}$
$-\ 3\ 6\ 4\ 6$

(e) $4^{1}3^{0}9^{1}2^{0}$
$-\ 3\ 6\ 4\ 6$

Begin by placing a zero above and to the right of the last digit in the minuend as in step (b). From this point on, the notation is completed by comparing corresponding digits. If the digit in the subtrahend is greater than the corresponding digit in the minuend, a "1" is entered to the left and above the digit in the minuend. If the digit is less than the corresponding digit in the minuend, a "0" is entered (steps c, d, e). The notation stops with the highest place value of the subtrahend.

An exception to this notation occurs at any point in the comparison process where the digits are the same. When this occurs, the notation entered for the last set of digits is again entered. In the following examples, this is illustrated.

(a) $4\ 3\ 6\ 9\ 2$
$-\ 3\ 4\ 6\ 8\ 7$

$4^{1}3^{0}6^{0}9^{1}2^{0}$
$-\ 3\ 4\ 6\ 8\ 7$
$1\ 9\ 0\ 1\ 5$

(b) $9\ 2\ 6\ 3\ 2$
$-\ 4\ 6\ 8\ 3\ 3$

$9^{1}2^{1}6\ {}^{1}3^{1}2^{0}$
$-\ 4\ 6\ 8\ \ 3\ 3$
$5\ 6\ 8(10)9$

Phase 2

Intermediate subtractions occur at this phase. Each digit in the subtrahend is subtracted from the corresponding digits in the minuend. Note that this may result in any entry of "10" in one place (see b). Remember that this is not the final answer.

Phase 3

The final phase of the algorithm involves subtraction of the notation in phase 1 from the intermediate subtractions of phase 2. This above problem would result in the following:

(a)
$$\begin{array}{r} 4^{1}3^{0}6^{0}9^{1}2^{0} \\ -\ 3\ 4\ 6\ 8\ 7 \\ \hline 1\ 9\ 0\ 1\ 5 \\ -\ 1\ 0\ 0\ 1\ 0 \\ \hline 9\ 0\ 0\ 5 \end{array}$$

(b)
$$\begin{array}{r} 9^{1}2^{1}6\ {}^{1}3^{1}2^{0} \\ -\ 4\ 6\ 8\ \ 3\ 3 \\ \hline 5\ 6\ 8(10)9 \\ -\ 1\ 1\ 1\ \ 1\ 0 \\ \hline 4\ 5\ 7\ 9\ 9 \end{array}$$

In both phase 1 and phase 3 of this problem, the order of the subtractions is unimportant. Cognitive load is decreased due to the subtraction of single-digit numbers.

This algorithm is a variation of the equal addends method. The use of the zero notation to the right of the ones place in the minuend, results in a units digit. In example b, the answer in the third line is therefore 11,110 more than the final answer. Thus, the subtraction of the "notation" results is the final answer.

Summary

Research has shown that different types of learners may profit from exploration of alternative algorithms. Although some students in the middle grades exhibit problems using the conventional algorithms for addition and subtraction, many teachers are not willing to explore these alternatives. The exploration of other algorithms can be a worthwhile experience for both the teacher and the student. It may lead to success in an area that has been frustrating for the student.

Pupils who have met success using conventional techniques will profit from the exploration into what makes these algorithms work. Extensions and enrichment activities can be developed to help these students gain a better understanding of our number system.

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